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JHEP11(2009)002

(http://iopscience.iop.org/1126-6708/2009/11/002)

Published by IOP Publishing for SISSA

Received: *August 31, 2009*
Accepted: *October 3, 2009*
Published: *November 4, 2009*

A_{N-1} conformal Toda field theory correlation functions from conformal $\mathcal{N} = 2$ SU(N) quiver gauge theories

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Abstract: We propose a relation between correlation functions in the $2d$ A_{N-1} conformal Toda theories and the Nekrasov instanton partition functions in certain conformal $\mathcal{N} = 2$ SU(N) $4d$ quiver gauge theories. Our proposal generalises the recently uncovered relation between the Liouville theory and SU(2) quivers [1]. New features appear in the analysis that have no counterparts in the Liouville case.

Keywords: Extended Supersymmetry, Duality in Gauge Field Theories, Conformal and W Symmetry

ArXiv ePrint: 0907.2189

doi:10.1088/1126-6708/2009/11/002

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Contents

1 Introduction

Last month Alday, Gaiotto and Tachikawa uncovered a remarkable connection between a class of conformal $\mathcal{N} = 2$ supersymmetric SU(2) quiver gauge theories and the conformal Liouville field theory in two dimensions [1].

The class of SU(2) quiver theories was introduced and studied in [2] (see also [3–5]) and comprise conformal $\mathcal{N} = 2$ theories whose gauge group is a product of SU(2) factors. The cases with matter in only fundamental, adjoint and bifundamental representations are quivers of conventional type, but the class of theories also contains more exotic possibilities (called generalized quivers in [2]). As emphasised in [2] it is convenient to focus on a maximal rank subgroup of the flavour symmetry group, composed only of SU(2) factors. The quiver can then be drawn in a way so that the SU(2) flavour factors correspond to external legs. When drawn in this manner the quivers resemble the diagrams associated with conformal blocks in $2d$ conformal field theory. In fact this is not a coincidence. In [1] it was argued that the Nekrasov instanton partition function associated with the gauge theory described by a certain quiver diagram is identical to the conformal block of the corresponding diagram in the Liouville field theory. Furthermore, the integral of the absolute value squared of the full partition function (including also the perturbative pieces) gives rise to a correlation function of primary fields in the Liouville theory. Note

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that in this proposal different correlation functions in *one* $2d$ CFT correspond to instanton partition functions in *different* $4d$ gauge theories.

A natural question to ask is if there are similar connections between the class of $4d$ SU(N) $\mathcal{N}=2$ quiver theories discussed in [2] and some class of $2d$ CFTs.

The two-dimensional A_{N-1} conformal Toda field theories are a generalisation of the Liouville model (which is identical to the A_1 Toda theory). In this paper we argue that there is a relation between the SU(N) quiver theories and the A_{N-1} Toda theories.

In the next section we briefly review the proposal in [1] focusing on those aspects that are most relevant for the extension to the class of SU(N) quiver theories. In section 3, we first review the $2d$ conformal Toda theories as well as the $4d$ quiver theories with SU(N) gauge groups and then make a proposal for how they are connected. We also perform a few tests of the suggested relations. The proposal shares many features with the Liouville case, but there are also new features on both the CFT and the gauge theory side. We conclude with a brief summary and discuss some open problems. Some more technical aspects have been relegated to an appendix.

2 Liouville & SU(2) quivers

In this section we review the proposal made in [1]. This is done in a way which makes it easy to highlight the differences between the Liouville theory and the Toda theories discussed in section 3.

2.1 The Liouville conformal field theory

The Liouville field theory is defined by the action

$$S = \int \mathrm{d}^2\sigma\sqrt{g}\left[\frac{1}{4\pi}g^{ad}\partial_a\phi\partial_d\phi + \mu\, e^{2b\phi} + \frac{Q}{4\pi}R\,\phi\right], \tag{2.1}$$

where g_{ad} $(a,d = 1,2)$ is the metric on the two-dimensional worldsheet, and R is its associated curvature. This theory is conformal provided Q and b are related via:

$$Q = b + \frac{1}{b}, \tag{2.2}$$

and the central charge of the theory is

$$c = 1 + 6\left(b + \frac{1}{b}\right)^2. \tag{2.3}$$

The Liouville theory has a set of primary fields

$$V_\alpha = e^{2\alpha\phi}, \tag{2.4}$$

with conformal dimension $\Delta(\alpha) = \alpha(Q-\alpha)$.

As is well known, the general form of the three-point correlation function in a $2d$ conformal field theory is

$$\langle V_{\alpha_1}(z_1,\bar{z}_1)V_{\alpha_2}(z_2,\bar{z}_2)V_{\alpha_3}(z_3,\bar{z}_3)\rangle = \frac{C(\alpha_1,\alpha_2,\alpha_3)}{|z_{12}|^{2(\Delta_1+\Delta_2-\Delta_3)}|z_{13}|^{2(\Delta_1+\Delta_3-\Delta_2)}|z_{23}|^{2(\Delta_2+\Delta_3-\Delta_1)}}. \tag{2.5}$$

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The correlation function of three primary fields in the Liouville theory was calculated in [6] (see also [7]) and takes the form

$$C(\alpha_1,\alpha_2,\alpha_3) = \left[\pi\mu\gamma(b^2)b^{2-2b^2}\right]^{\frac{Q-\alpha_1-\alpha_2-\alpha_3}{b}} \times \frac{\Upsilon(b)\Upsilon(2\alpha_1)\Upsilon(2\alpha_2)\Upsilon(2\alpha_3)}{\Upsilon(\alpha_1+\alpha_2+\alpha_3-Q)\Upsilon(-\alpha_1+\alpha_2+\alpha_3)\Upsilon(\alpha_1-\alpha_2+\alpha_3)\Upsilon(\alpha_1+\alpha_2-\alpha_3)}\,. \tag{2.6}$$

The function $\Upsilon(x)$ is defined as follows (note that $\Upsilon(x)$ depends on b even though this is not indicated explicitly)

$$\Upsilon(x) = \frac{1}{\Gamma_2(x|b,b^{-1})\Gamma_2(Q-x|b,b^{-1})}\,, \tag{2.7}$$

where $\Gamma_2(x|\epsilon_1,\epsilon_2)$ is defined via the relations

$$\log\Gamma_2(x|\epsilon_1,\epsilon_2) = \frac{\mathrm{d}}{\mathrm{d}s}\zeta_2(s,x|\epsilon_1,\epsilon_2)\Big|_{s=0}\,, \tag{2.8}$$

and

$$\zeta_2(s,x|\epsilon_1,\epsilon_2) = \sum_{m,n}\frac{1}{(m\epsilon_1+n\epsilon_2+x)^s} = \frac{1}{\Gamma(s)}\int \mathrm{d}t\, t^{s-1}\frac{e^{-tx}}{(1-e^{-\epsilon_1 t})(1-e^{-\epsilon_2 t})}\,. \tag{2.9}$$

The function $\Gamma_2(x|\epsilon_1,\epsilon_2)$ satisfies the identity

$$\Gamma_2(x+\epsilon_1|\epsilon_1,\epsilon_2)\,\Gamma_2(x+\epsilon_2|\epsilon_1,\epsilon_2) = x\,\Gamma_2(x|\epsilon_1,\epsilon_2)\,\Gamma_2(x+\epsilon_1+\epsilon_2|\epsilon_1,\epsilon_2)\,. \tag{2.10}$$

Finally,

$$\gamma(x) = \frac{\Gamma(x)}{\Gamma(1-x)}\,. \tag{2.11}$$

where $\Gamma(x)$ is the ordinary gamma function. The functions $\Upsilon(x)$ and $\gamma(x)$ are related via (note that $\Upsilon(Q-x)=\Upsilon(x)$ and $\Upsilon(b)=\Upsilon(\frac{1}{b})=\Upsilon'(0)$)

$$\begin{aligned}\Upsilon(x+b) &= \gamma(bx)\,b^{1-bx}\,\Upsilon(x)\,,\\ \Upsilon\left(x+\frac{1}{b}\right) &= \gamma\left(\frac{x}{b}\right)b^{2x/b-1}\,\Upsilon(x)\,.\end{aligned} \tag{2.12}$$

Higher-point correlation functions in any CFT can be related to the three-point function of primary fields, which therefore determines the entire theory [8]. (In subsequent formulæ we suppress the dependence on the antiholomorphic coordinates.) The general form of the four-point function is

$$\langle V_{\alpha_1}(z_1)V_{\alpha_2}(z_2)V_{\alpha_3}(z_3)V_{\alpha_4}(z_4)\rangle = \left(\prod_{i<j}z_{ij}^{\delta_{ij}}\right)F_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}(z)\,. \tag{2.13}$$

Here $\delta_{ij}=\delta_{ji}$ is any solution of $\sum_{i\neq j}\delta_{ij}=2\Delta_j$, $z_{ij}=z_i-z_j$ and $z=z_{12}z_{34}/z_{23}z_{41}$ is the cross ratio. Different choices of δ_{ij} can be absorbed in a redefinition of $F_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}(z)$. It is convenient to fix three points to $0,1,\infty$ and define

$$\begin{aligned}\langle V_{\alpha_1}(0)V_{\alpha_2}(1)V_{\alpha_3}(z)V_{\alpha_4}(\infty)\rangle &= \lim_{z_4\to\infty} z_4^{2\Delta_4}\langle V_{\alpha_1}(0)V_{\alpha_2}(1)V_{\alpha_3}(z)V_{\alpha_4}(z_4)\rangle\\ &= z^{\delta_{12}}(1-z)^{\delta_{13}}F_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}(z)\,.\end{aligned} \tag{2.14}$$

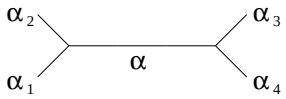

Figure 1. Conformal block of four-point function in the s-channel.

In the Liouville theory we use a bra-ket notation, which has the property $\langle\alpha|\alpha\rangle = 1$, and is such that

$$\langle\alpha_1|V_{\alpha_2}(1)V_{\alpha_3}(z)|\alpha_4\rangle = \langle V_{Q-\alpha_1}(0)V_{\alpha_2}(1)V_{\alpha_3}(z)V_{\alpha_4}(\infty)\rangle\,. \tag{2.15}$$

Inserting a complete set of states we find

$$\langle\alpha_1|V_{\alpha_2}(1)V_{\alpha_3}(z)|\alpha_4\rangle = \int \mathrm{d}\alpha \sum_{\mathbf{k},\mathbf{k}'} \langle\alpha_1|V_{\alpha_2}(1)|\psi_{\mathbf{k}}(\alpha)\rangle (K^{-1})_{\mathbf{k},\mathbf{k}'} \langle\psi_{\mathbf{k}'}(\alpha)|V_{\alpha_3}(z)|\alpha_4\rangle\,. \tag{2.16}$$

Here the intermediate states $|\psi_{\mathbf{k}}(\alpha)\rangle$ are descendants of the primary state labelled by α, i.e.

$$|\psi_{-\mathbf{k}}(\alpha)\rangle \equiv L_{-\mathbf{k}}|\alpha\rangle = L_{-k_1}\cdots L_{-k_p}|\alpha\rangle\,, \tag{2.17}$$

and $\mathbf{k} = (k_1, \ldots, k_p)$ is a partition of $k \equiv |\mathbf{k}|$ i.e. $\sum_{i=1}^{p} k_i = k$. Finally, K is the Gram matrix defined as

$$K = \langle\psi_{-\mathbf{k}}|\psi_{-\mathbf{k}'}\rangle\,. \tag{2.18}$$

A special situation arises if one of the α_i corresponds to a degenerate state i.e. a state that is annihilated by some combination of $L_{\mathbf{k}}$'s. In such a situation there will be restrictions on the allowed intermediate states, and it may be that only a discrete number is allowed.

The above expression can be represented graphically as in figure 1.

Note that (2.16) can be calculated perturbatively using the commutation relations between the L_k's and $V_\alpha(z)$:[1]

$$\begin{aligned}[L_m, V_\alpha] &= z^m[\,(m+1)\,\Delta(\alpha)\,V_\alpha + z\,(L_{-1}V_\alpha)\,] \\ &= z^m(\,m\,\Delta(\alpha)\,V_\alpha + [L_0, V_\alpha]\,)\,.\end{aligned} \tag{2.19}$$

It is easy to see that $\langle\psi_{\mathbf{k}'}(\alpha)|V_{\alpha_3}(z)|\alpha_4\rangle$ is proportional to $\langle\alpha|V_{\alpha_3}(z)|\alpha_4\rangle$. Therefore (2.16) can in principle be determined [8]. The ratio

$$\frac{\sum_{\mathbf{k},\mathbf{k}'}\langle\alpha_1|V_{\alpha_2}(1)|\psi_{\mathbf{k}}(\alpha)\rangle (K^{-1})_{\mathbf{k},\mathbf{k}'} \langle\psi_{\mathbf{k}'}(\alpha)|V_{\alpha_3}(z)|\alpha_4\rangle}{\langle\alpha_1|V_{\alpha_2}(1)|\alpha\rangle\langle\alpha|V_{\alpha_3}(z)|\alpha_4\rangle} \tag{2.20}$$

is called a conformal block. General n-point functions can be dealt with in an analogous manner. They depend on $n-3$ cross ratios.

[1] Here $(L_{-1}V_\alpha)$ is the field obtained by acting with L_{-1} on V_α. Using the known properties of the Virasoro generators gives $(L_{-1}V_\alpha) = \partial V_\alpha$.

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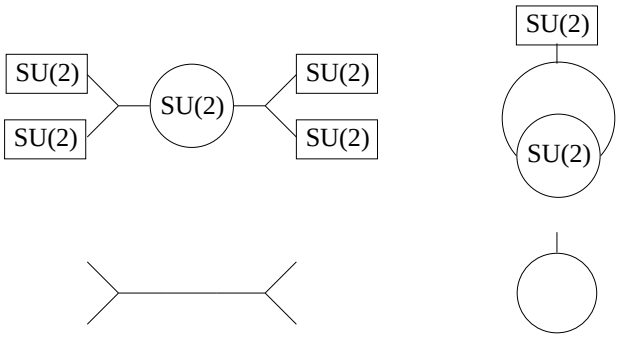

Figure 2. Quiver diagrams for the two simplest SU(2) quivers.

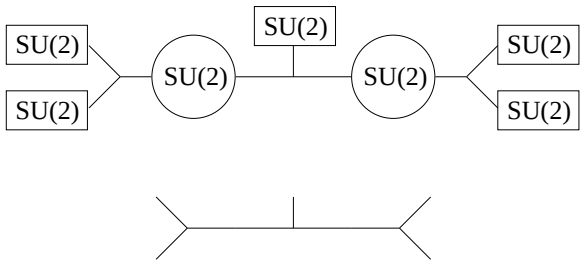

Figure 3. Another example of an SU(2) quiver.

2.2 The conformal $\mathcal{N} = 2$ SU(2) quiver theories

The class of $\mathcal{N} = 2$ SU(2) quiver gauge theories introduced and studied in [2] have matter (flavour) fields in various representations. Let us recall the following general results: if we have n equal, pseudoreal representations then the global flavour symmetry group contains an SO($2n$) factor; if we have n equal, real representations then we get an Sp($2n$) flavour factor; and if the equal representations are complex we find U(n) flavour symmetry.

To get a conformal theory we need to have a suitable matter content to get a vanishing β-function. If the gauge group is a single SU(2), we can get a conformal theory from the following matter content: either 4 fundamentals or one adjoint. As the fundamental representation of SU(2) is pseudoreal the first theory has an SO(8) flavour symmetry which has an $\mathrm{SO}(4)^2 = \mathrm{SU}(2)^4$ subgroup. The adjoint is a real representation so the flavour symmetry of the second theory is $\mathrm{Sp}(2) = \mathrm{SU}(2)$. The resulting two theories can be illustrated graphically in terms of quiver diagrams as in figure 2.

Here boxed SU(2)'s correspond to SU(2) factors in the flavour symmetry group, whereas circled SU(2)'s correspond to gauge groups. Simplifying the quiver diagrams by stripping off the boxes and circles we find the diagrams on the second line of the above figure.

We can also consider gauge theories with a product gauge group. In the figure below we draw the quiver diagram for the SU(2)×SU(2) theory with two matter fields in the fundamental representation of each factor of the gauge group, and one matter field in the bifundamental representation (which is a real representation), as in figure 3.

The theories we considered above were all conventional quivers, albeit drawn in a slightly unconventional manner. An important insight in [2] was that these theories belong

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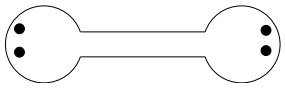

Figure 4. A particular degeneration of the sphere with four punctures.

to a larger class of theories, denoted $\mathcal{T}_{(n,g)}(A_1)$. The theories in this larger class can be viewed as the arising from the six-dimensional A_1 (2,0) theory [9] compactified on $C \times \mathbb{R}^4$ where C is a genus g Riemann surface with n punctures. The punctures arise from n codimension 2 defects of a certain type which fill $\mathbb{R}^4$ and intersect C at points. From the gauge theory perspective, the punctures correspond to the SU(2) factors in the flavour symmetry group, and the genus of the Riemann surface depends on the number of loops in the quiver diagram. It is the A_1 theory that is relevant since the gauge group contains SU(2) factors.

Consider the SU(2)+4$\square$ theory discussed above. The Riemann surface in this case is a sphere with four punctures. When this Riemann surface is deformed into two spheres with two punctures each connected by a thin tube as in the figure below we obtain the weakly coupled gauge theory description (note the obvious similarity with the quiver diagram), as in figure 4.

In the second example above (SU(2) with adjoint matter) the Riemann surface is a torus with one puncture.

There are many more things that could be discussed. Here we only very briefly mention a few further salient points and refer to the original paper [2] for more details.

- The rules for drawing quiver diagrams corresponding to theories in $\mathcal{T}_{(n,g)}(A_1)$ also involve an internal three-point vertex. Quivers involving such vertices were called generalised quivers in [2].

- S-duality was an important guideline in the construction of the $\mathcal{T}_{(n,g)}(A_1)$ theories. An S-duality transformation in general changes the quiver diagram in a non-trivial way.

- The Riemann surface C is not quite the Seiberg-Witten curve. The Seiberg-Witten curve was constructed in [2] (drawing on earlier work [10]) and is a certain branched covering of C. The Seiberg-Witten differential was also written down. Both the Seiberg-Witten curve and differential were written in a way which clarified the six-dimensional origin of the class of theories.

- The gauge coupling constants are associated with sewing parameters, q_i, of the tubular regions via $q_i = e^{2\pi i \tau_i}$ where τ_i is the gauge coupling of ith factor of the gauge group (more precisely the τ that appears here is $\tau_{\rm UV}$, cf. [1, 11]).

2.3 The relation between the two

We saw above that the SU(2) quiver diagrams when drawn in a particular way (cf. figures 2, 3) look like the diagrams associated with the conformal blocks in a $2d$ conformal field

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theory (cf. figure 1). In [1] this resemblance was turned into a concrete proposal relating the SU(2) quiver theories to a particular conformal field theory — the Liouville theory.

In an $\mathcal{N} = 2$ gauge theory the natural basic object to consider is the prepotential. This is most efficiently determined using the instanton counting method of Nekrasov [12, 13]. In this approach one introduces a deformation of the theory with two parameters ϵ_1 and ϵ_2 which belong to an SO(2)×SO(2) subgroup of the SO(4) Lorentz symmetry. The power of this deformation is that it ensures that the integrals over instanton moduli space localise to points; the integrals can therefore be performed in an algorithmic manner. The fundamental object in Nekrasov's approach is the partition function[2] $Z(a, \mu_j, \epsilon_1, \epsilon_2, q)$ where a parameterise the Coulomb branch, the μ_j are the masses of the matter fields and $q = e^{2\pi i \tau}$ (since we consider a conformal theory the instanton expansion is in terms q and not Λ). The partition function factorises into two parts as

$$Z = Z_{\text{pert}}\, Z_{\text{inst}}\,, \tag{2.21}$$

where Z_{pert} is the contribution from perturbative calculations (because of supersymmetry there are contributions only at tree- and 1-loop-level), and Z_{inst} is the contribution from the instantons. The instanton part can be expanded as (k is the instanton number)

$$Z_{\text{inst}} = \sum_k q^k Z_k\,. \tag{2.22}$$

We should stress that one needs the theory to be weakly coupled to be able to apply the instanton counting method. The $\mathcal{N} = 2$ prepotential $\mathcal{F}$ is recovered from Z when $\epsilon_1 = -\epsilon_2 = \hbar$ via the following formula (in the limit $\hbar \to 0$)

$$Z = e^{-\frac{1}{\hbar^2}\mathcal{F}}\,, \tag{2.23}$$

although in this paper it is Z rather than $\mathcal{F}$ that will be important.

The proposal in [1] is as follows. The instanton partition function (2.22) of a certain quiver diagram is conjectured to be equal to the conformal block in the Liouville theory corresponding to the same diagram. The relations between the parameters on the two sides are

$$\epsilon_1 = b\,, \qquad \epsilon_2 = \frac{1}{b}\,, \tag{2.24}$$

the q_i are identified with the cross-ratios z_i and the masses m_i associated with the various SU(2) factors of the flavour symmetry essentially correspond to the external α_i in the Liouville theory. The a's of the SU(2) gauge factors essentially correspond to the internal α's in the conformal block. (The precise rules will be illustrated in a simple example below).

We saw above that a quiver diagram with n external legs corresponds to a particular degeneration of a Riemann surface with n punctures. In the conformal block this corresponds to choosing a specific intermediate channel (e.g. the s-channel as in figure 1). Other channels in the conformal block correspond to other degenerations of the Riemann surface.

[2] For simplicity we focus on a quiver with a single gauge group and hence one a and one q. Some further information can be found in appendix A.1.

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The various possibilities are related by the crossing symmetry on the CFT side and by S-duality on the gauge theory side [1]. Note that the identification (2.24) implies that one can not set both ϵ_i to zero (since $\epsilon_1\epsilon_2$=1); the deformation on the gauge theory side is therefore essential.

A further proposal was also made in [1]:[3] the absolute value squared of the complete partition function (2.21) integrated over the a's of the SU(2) gauge factors with the natural measure should correspond to the full Liouville correlation function. Essentially what happens is that the perturbative contributions combine to give the three-point function factors and the instantons give the conformal block.

A simple example will illustrate the proposal. Consider the SU(2) + 4□ theory (the first example in figure 2). For this theory the Z_k can be calculated in two ways. One can either view SU(2) as a restriction of U(2) or as Sp(2). In the latter case one can use the results for Sp($2n$) instanton counting obtained in [15, 16]. The leading results obtained using the two methods are listed in (A.3), (A.11). We find that the two expressions are related as

$$Z_{\text{inst}}^{\text{SU(2)}\subset\text{U(2)}}\left(a, m+\frac{\epsilon}{2}, \kappa-\epsilon\right) = (1-q)^{Q_2(a,m,\kappa,\epsilon)} Z_{\text{inst}}^{\text{SU(2)=Sp(2)}}(a,m,\kappa)\,, \tag{2.25}$$

where $Q_2(a,m,\kappa,\epsilon)$ is the quadratic expression

$$Q_2(a,m,\kappa,\epsilon) = \frac{1}{2}(a^2 - m_1^2 - m_2^2 + \kappa_1^2 + \kappa_2^2 + 4\kappa_1\kappa_2 - 2\kappa_1\epsilon - 2\kappa_2\epsilon) - \frac{3}{8}\epsilon^2\,. \tag{2.26}$$

In [1] it was argued that one should factor out an overall $(1-q)^{Q_2'(\kappa,\epsilon)}$ from $Z_{\text{inst}}^{\text{SU(2)}\subset\text{U(2)}}$ when comparing with the CFT conformal block. Here Q_2' is a certain quadratic expression, different from Q_2, that only depends on κ_i and ϵ. It was argued that this prefactor was due to an incomplete decoupling of the U(1) inside U(2). The prepotential of an $\mathcal{N}=2$ gauge theory is only determined up to a quadratic expression, which at the level of the partition function translates into the statement that $\log Z$ is only defined up to a quadratic expression. We see that the different prefactors are examples of this ambiguity and therefore do not lead to a contradiction. Since the prefactors always seem to involve $(1-q)$ it is not inconceivable that the different calculations correspond to different choices of the δ_{ij} in (2.14).

The prefactor one has to factor out from $Z_{\text{inst}}^{\text{SU(2)}\subset\text{U(2)}}$ to get agreement with the conformal block calculation is closely related to U(1) in the following sense. If we directly calculate the partition function for a U(1) theory coupled to a fundamental and an anti-fundamental matter field we find

$$Z_{\text{inst}} = \sum_{Y_{\mathbf{k}}} q^{|\mathbf{k}|} \prod_{\text{boxes}\in Y_{\mathbf{k}}} \frac{(\epsilon_1\, i + \epsilon_2\, j - \mu)(\epsilon_1\, i + \epsilon_2\, j + \bar{\mu} - \epsilon)}{(\epsilon_2(A{+}1) - \epsilon_1 L)(\epsilon - \epsilon_2(A{+}1) + \epsilon_1 L)} = (1-q)^{\frac{(\mu-\epsilon)\,\bar{\mu}}{\epsilon_1\epsilon_2}} \tag{2.27}$$

where the sum runs over all Young diagrams and the product is over all boxes in the diagram; (i,j) label the columns and rows and A and L are the arm and leg lengths of a box. A similar calculation for the second model in figure 2 gives (cf. (6.12) in [13])

$$\sum_k q^k \prod_{\text{boxes}} \frac{(\epsilon_2(A{+}1) - \epsilon_1 L - \mu)(\epsilon - \epsilon_2(A{+}1) + \epsilon_1 L - \mu)}{(\epsilon_2(A{+}1) - \epsilon_1 L)(\epsilon - \epsilon_2(A{+}1) + \epsilon_1 L)} = \prod_{n=1}^{\infty}(1-q^n)^{\frac{\mu\,(\epsilon-\mu)}{\epsilon_1\epsilon_2}-1}\,. \tag{2.28}$$

[3] This proposal was inspired by the earlier work [14].

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The above two expressions (with $\epsilon_1\epsilon_2 = 1$) are similar to (C.1), (C.5) in [1].

Continuing with our example, we calculate the first non-trivial term in the conformal block as

$$\frac{\langle\alpha_1|V_{\alpha_2}(1)L_{-1}|\alpha\rangle(\langle\alpha|L_1L_{-1}|\alpha\rangle)^{-1}\langle\alpha|L_1V_{\alpha_3}(z)|\alpha_4\rangle}{\langle\alpha_1|V_{\alpha_2}(1)|\alpha\rangle\langle\alpha|V_{\alpha_3}(z)|\alpha_4\rangle} = z\,\frac{[\Delta(\alpha)+\Delta(\alpha_2)-\Delta(\alpha_1)][\Delta(\alpha)+\Delta(\alpha_3)-\Delta(\alpha_4)]}{2\Delta(\alpha)}\,. \tag{2.29}$$

Comparing this expression to the one-instanton calculation in the SU(2) theory (A.3) we find agreement provided we factor out $(1-q)^{(2\kappa_1+Q))(2\kappa_2+Q)/2}$ from Z and identify (in our conventions)

$$z = -q\,,\quad m_1 = \alpha_1\,,\quad m_2 = \alpha_4\,,\quad \kappa_1 = -\alpha_2 - Q/2\,,\quad \kappa_2 = -\alpha_3 - Q/2\,,\quad a = \alpha - Q/2\,. \tag{2.30}$$

Furthermore, using (A.6) we see that the perturbative contribution to the partition function is

$$\begin{aligned} Z_{\rm pert} = {} & \exp\left[-\gamma_{\epsilon_1\epsilon_2}(2a-\epsilon_1) - \gamma_{\epsilon_1\epsilon_2}(2a-\epsilon_2) + \gamma_{\epsilon_1\epsilon_2}(-a-m_1-\kappa_1)\right] \\ & \cdot\exp\left[\gamma_{\epsilon_1\epsilon_2}(-a-m_2-\kappa_2)+\gamma_{\epsilon_1\epsilon_2}(-a+m_1-\kappa_1-\epsilon)+\gamma_{\epsilon_1\epsilon_2}(-a+m_2-\kappa_2-\epsilon)\right] \\ & \cdot\exp\left[\gamma_{\epsilon_1\epsilon_2}(a-m_1-\kappa_1)+\gamma_{\epsilon_1\epsilon_2}(a-m_2-\kappa_2)+\gamma_{\epsilon_1\epsilon_2}(a+m_1-\kappa_1-\epsilon)+\gamma_{\epsilon_1\epsilon_2}(a+m_2-\kappa_2-\epsilon)\right] \end{aligned} \tag{2.31}$$

whose absolute value squared should be compared to (via (A.7), (2.7) and the identifications (2.24) and (2.30))

$$\begin{aligned} \langle\alpha_1|V_{\alpha_2}(1)|\alpha\rangle\langle\alpha|V_{\alpha_3}(z)|\alpha_4\rangle = {} & \left[\pi\mu\gamma(b^2)b^{2-2b^2}\right]^{(\alpha_1-\alpha_2-\alpha_3-\alpha_4)/b} \\ & \times\frac{\Upsilon(b)\Upsilon(2(Q-\alpha_1))\Upsilon(2\alpha_2)\Upsilon(2\alpha)}{\Upsilon(-\alpha_1+\alpha_2+\alpha)\Upsilon(Q-\alpha_1+\alpha_2-\alpha)\Upsilon(Q-\alpha_1-\alpha_2+\alpha)\Upsilon(-Q+\alpha_1+\alpha_2+\alpha)} \\ & \times\frac{\Upsilon(b)\Upsilon(2(Q-\alpha))\Upsilon(2\alpha_3)\Upsilon(2\alpha_4)}{\Upsilon(-\alpha+\alpha_3+\alpha_4)\Upsilon(Q-\alpha+\alpha_3-\alpha_4)\Upsilon(Q-\alpha-\alpha_3+\alpha_4)\Upsilon(-Q+\alpha+\alpha_3+\alpha_4)}. \end{aligned} \tag{2.32}$$

If we redefine the primary fields V_α by introducing

$$\mathcal{V}_\alpha = \frac{\left[\pi\mu\gamma(b^2)b^{2-2b^2}\right]^{\alpha/b}}{\Upsilon(2\alpha)}V_\alpha\,, \tag{2.33}$$

and use the identity (2.10) most of the differences between $\int \mathrm{d}a\,(2a)^2\,|Z|^2$ and the four-point correlation function can be removed. The remaining discrepancy is a factor depending only on b, as well as the q and $(1-q)$ pieces. It does not seem unlikely that these remaining differences can also be removed by a proper definition of Z. In this context we note that the $(1-q)$ factors one naturally gets in the instanton calculations are not unsimilar to those that have appeared in the Liouville literature, see e.g. (2.7) in [17]. Factors of $q^{Q_2''}$ can be obtained from (unphysical) $2\pi i\tau Q_2''$ pieces in the prepotential.

3 Toda & SU(N) quivers

In this section we first review the conformal A_{N-1} Toda theories and then review the class of SU(N) quiver theories. Finally, we make a proposal for how the two are related.

3.1 The A_{N-1} conformal Toda field theories

The A_{N-1} Toda field theories are defined by the action

$$S = \int \mathrm{d}^2\sigma\sqrt{g}\left[\frac{1}{8\pi}g^{ad}\langle\partial_a\phi,\partial_d\phi\rangle + \mu\sum_{i=1}^{N-1}e^{b\langle e_i,\phi\rangle} + \frac{\langle Q,\phi\rangle}{4\pi}R\,\phi\right], \tag{3.1}$$

where g_{ad} $(a,d = 1,2)$ is the metric on the two-dimensional worldsheet, and R is its associated curvature. Furthermore, the e_i are the simple roots of the A_{N-1} Lie algebra, $\langle\cdot,\cdot\rangle$ denotes the scalar product on the root space, and the $(N-1)$-dimensional vector of fields ϕ can be expanded as $\phi = \sum_i \phi_i e_i$.

From the above action it is easy to see that the Liouville theory (2.1) is identical to the A_1 Toda field theory.[4] The Toda theories with rank >1 are much more complicated than the Liouville model. Toda theories can be defined for any simple Lie algebra by taking the e_i to be the simple roots of the corresponding Lie algebra.[5]

The A_{N-1} Toda theory is conformal provided Q and b are related via:

$$Q = \left(b + \frac{1}{b}\right)\rho\,, \tag{3.2}$$

where ρ is the Weyl vector (half the sum of all positive roots). The central charge is [18][6]

$$c = N - 1 + 12\langle Q,Q\rangle = (N-1)\left(1 + N(N+1)\left(b+\frac{1}{b}\right)^2\right), \tag{3.3}$$

where in the second step we used the Freudenthal-de Vries strange formula.

For $N > 2$ the symmetry algebra of the A_{N-1} Toda theories contains in addition to the stress tensor, T, also $N-2$ additional holomorphic symmetry currents with conformal dimensions $3,4,\ldots,N+1$ [19]. These symmetry currents are usually denoted $\mathcal{W}^{(k)}$ $(k = 2,\ldots,N+1)$, where $\mathcal{W}^{(2)} \equiv T$, and form a $\mathcal{W}_{N+1}$-algebra (see e.g. [20] for a review of $\mathcal{W}$-algebras). As an example, in the A_2 Toda theory there is only a single extra current, $\mathcal{W}^{(3)}(z)$. Together with the stress tensor it forms the so called $\mathcal{W}_3$ algebra [21]. The mode expansions of the currents are

$$T(z) = \sum_n L_n z^{-n-2}\,, \qquad \mathcal{W}^{(3)}(z) = \sum_n W_n^{(3)} z^{-n-3} \equiv \sum_n W_n z^{-n-3}\,. \tag{3.4}$$

[4] Because of the standard Lie algebra conventions, some formulæ differ at first sight.

[5] Toda theories can also be defined for affine Lie algebras by adding an additional simple root. The resulting theories are non-conformal and will play no role in this paper.

[6] As written this formula is true only for the A_{N-1} Lie algebras. In the general case (including also non-simply laced cases) the formula gets replaced by $c = r + 12\langle Q,Q\rangle$ where r is the rank and $Q = b\rho + \frac{\rho^\vee}{b}$ where $\rho^\vee$ is the dual Weyl vector (half the sum of the positive coroots). Note the duality under $b \leftrightarrow \frac{1}{b}$ and $\rho \leftrightarrow \rho^\vee$.

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In terms of the modes (3.4) the $\mathcal{W}_3$ algebra can be written

$$\begin{aligned}
[L_n, L_m] &= (n-m)L_{n+m} + \frac{c}{12}(n^3-n)\delta_{n,-m}\,, \\
[L_n, W_m] &= (2n-m)W_{n+m}\,, \\
[W_n, W_m] &= \frac{c}{3\cdot 5!}(n^2-1)(n^2-4)n\delta_{n,-m} + \frac{16}{22+5c}(n-m)\Lambda_{n+m} \\
&\quad +(n-m)\left(\frac{1}{15}(n+m+2)(n+m+3) - \frac{1}{6}(n+2)(m+2)\right)L_{n+m}\,,
\end{aligned} \tag{3.5}$$

where c is the central charge (3.3) and

$$\Lambda_n = \sum_{k=-\infty}^{\infty} :L_k L_{n-k}: + \frac{1}{5}x_n L_n\,, \tag{3.6}$$

with

$$x_{2l} = (1+l)(1-l)\,, \qquad x_{2l+1} = (2+l)(1-l)\,. \tag{3.7}$$

Primary fields can be defined in analogy with the Virasoro case. A $\mathcal{W}$-primary field satisfies

$$W_0^{(k)}V = w^{(k)}V\,, \qquad W_n^{(k)}V = 0 \quad \text{when } n>0\,. \tag{3.8}$$

In the Toda theories the ($\mathcal{W}$) primary fields are

$$V_\alpha = e^{\langle \alpha,\phi\rangle}\,. \tag{3.9}$$

In the particular example of the A_2 theory the primary fields satisfy

$$L_0V_\alpha = \Delta(\alpha)V_\alpha\,, \qquad W_0V_\alpha = w(\alpha)V_\alpha\,, \qquad L_nV_\alpha = W_nV_\alpha = 0 \quad \text{when } n>0\,, \tag{3.10}$$

where

$$\Delta(\alpha) = \frac{(2Q-\alpha,\alpha)}{2}\,, \tag{3.11}$$

is the conformal dimension and

$$w(\alpha) = i\sqrt{\frac{48}{22+5c}}\,(\alpha-Q,\lambda_1)(\alpha-Q,\lambda_2)(\alpha-Q,\lambda_3)\,, \tag{3.12}$$

is the quantum number of the $\mathcal{W}^{(3)}$ current (here the λ_i are the weights of the fundamental representation of the A_2 Lie algebra).

The three-point function of primary fields is defined as in (2.5). Except in the A_1 case, only partial results are known. Recently it was shown [22, 23] that in the special case when one of the α_i's takes one of the two special values

$$\alpha = \varkappa\omega_1\,, \quad \text{or} \qquad \alpha = \varkappa\omega_{N-1}\,, \tag{3.13}$$

where ω_1 (ω_{N-1}) is the highest weight of the fundamental (antifundamental) representation of the A_{N-1} Lie algebra, the three-point function is given by (2.5) with

$$\begin{aligned}
C(\alpha_1,\alpha_2,\varkappa\omega_{N-1}) &= \left[\pi\mu\gamma(b^2)b^{2-2b^2}\right]^{\frac{\langle 2Q-\sum\alpha_i,\rho\rangle}{b}} \\
&\quad\times\frac{(\Upsilon(b))^{N-1}\,\Upsilon(\varkappa)\prod_{e>0}\Upsilon\big(\langle Q-\alpha_1,e\rangle\big)\Upsilon\big(\langle Q-\alpha_2,e\rangle\big)}{\prod_{ij}\Upsilon\big(\frac{\varkappa}{n}+\langle\alpha_1-Q,\lambda_i\rangle+\langle\alpha_2-Q,\lambda_j\rangle\big)}\,,
\end{aligned} \tag{3.14}$$

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where the product in the numerator is over all positive roots and in the denominator the λ_i are the weights of the representation with highest weight ω_{N-1}. (The result for $\alpha_3 = \varkappa\omega_1$ is obtained by replacing λ_i by $\lambda'_i = -\lambda_{N+1-i}$.)

We now come to a crucial difference with the Liouville case. It is no longer true that the higher-point functions of $\mathcal{W}$ primary fields are determined in terms of the three-point function of $\mathcal{W}$ primary fields [24]. If one forgets about the full $\mathcal{W}$-algebra, it is of course still true that the higher-point functions are determined in terms of three-point functions of the *Virasoro* primaries (a larger set than the $\mathcal{W}$ primaries). However, the $\mathcal{W}$ symmetry, while constraining, is not powerful enough to determine the higher-point functions of $\mathcal{W}$ primaries in terms of the three-point functions of $\mathcal{W}$ primaries. To illustrate why this is so we focus on the A_2 theory. As in (2.16), the four-point function of ($\mathcal{W}$) primaries (3.9) can be decomposed by inserting a complete set of intermediate (descendant) states, which now are given by

$$|\psi_{(\mathbf{k},\mathbf{l})}(\alpha)\rangle \equiv L_{-\mathbf{k}}W_{-\mathbf{l}}|\alpha\rangle = L_{-k_1}\cdots L_{-k_p}W_{l_1}\cdots W_{-l_q}|\alpha\rangle\,. \tag{3.15}$$

Using the commutation relations

$$\begin{aligned}[L_m, V_\alpha] &= z^m[\,(m+1)\,\Delta(\alpha)\,V_\alpha + z\,(L_{-1}V_\alpha)\,]\,, \\ [W_m, V_\alpha] &= z^m[\,\frac{(m+1)(m+2)}{2}\,w(\alpha)\,V_\alpha + z\,(m+1)\,(W_{-1}V_\alpha) + z^2\,(W_{-2}V_\alpha)\,]\,,\end{aligned} \tag{3.16}$$

one can in general only reduce $\langle\psi_{\mathbf{k}'}(\alpha)|V_{\alpha_3}(z)|\alpha_4\rangle$ to belong to the set of states

$$\langle\alpha|V_{\alpha_3}(z)(W_{-1})^n|\alpha_4\rangle\,, \tag{3.17}$$

where n is any positive integer (including zero). It is therefore not true that all correlations functions are determined solely in terms of the three-point functions of $\mathcal{W}$ primary fields (see [24] for more details).

However, if some of the α_i take special values then the corresponding states may be semi-degenerate i.e. $(W_{-1})^p|\alpha_i\rangle$ can be expressed in terms of states of the form $L_{\mathbf{k}}(W_{-1})^n|\alpha_i\rangle$ with $n < p$. In such a situation, one can reduce any $\langle\psi_{\mathbf{k}'}(\alpha)|V_{\alpha_3}(z)|\alpha_4\rangle$ to belong to the set (3.17) with $n < p$. In the particular case when $p = 1$ the class of higher-point correlation functions are therefore determined in terms of three-point functions of $\mathcal{W}$ primary fields.

Continuing with the $\mathcal{W}_3$ example, it is known that [25]

$$\left(W_{-1} - \frac{3w(\alpha)}{2\Delta(\alpha)}L_{-1}\right)|\alpha\rangle = 0\,, \tag{3.18}$$

provided that α takes one of the two values (3.13) with $N = 3$. The corresponding state is therefore (semi-)degenerate. Note that the condition (3.18) implies (by acting with W_1)

$$\Delta_\alpha\left[\frac{32}{22+5c}\left(\Delta_\alpha + \frac{1}{5}\right) - \frac{1}{5}\right] = \frac{9}{2}\frac{w_\alpha^2}{\Delta_\alpha}\,. \tag{3.19}$$

Denoting the primaries corresponding to states satisfying (3.18) as $V'_\varkappa$, it follows from the above discussion that n-point functions of the form[7]

$$\langle\alpha_1|V'_{\varkappa_2}\cdots V'_{\varkappa_{n-1}}|\alpha_n\rangle\,, \tag{3.20}$$

[7]Here $\langle\alpha_1|\mathcal{O}|\alpha_n\rangle \equiv \langle V_{2Q-\alpha_1}\,\mathcal{O}\,V_{\alpha_n}\rangle$.

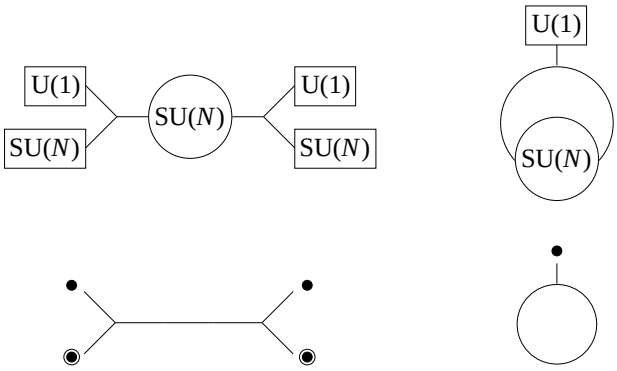

Figure 5. The two simplest examples of SU(N) quivers.

can be calculated perturbatively in terms of $\mathcal{W}$ chiral blocks (analogous to the Virasoro conformal blocks) and the special class of three-point functions of primary fields given in (3.14).

3.2 The conformal $\mathcal{N} = 2$ SU(N) quiver theories

We now turn to the class of conformal $\mathcal{N} = 2$ $4d$ SU(N) quiver theories. As in the SU(2) case discussed in section 2.2 we start with some simple examples. Theories with a single SU(N) gauge factor are conformal if the matter content comprise either $2N$ fundamentals or one adjoint. In the first case, the flavour symmetry group is U($2N$) since the fundamental representation is complex (for $N > 2$). It is convenient to focus on a U(N)×U(N) $\cong$ U(1)^{2}SU(N)^2 subgroup of U($2N$). The adjoint representation is complex so the flavour symmetry of the second model is U(1). For these models we can draw the quiver diagrams, as in figure 5.

In figure 5, the boxes refer to factors of the flavour symmetry group and the circles to the SU(N) gauge group. There are two different types of flavour symmetry factors in the above diagrams in contrast to the the SU(2) case where there was only one. In the second line of the above figure we have stripped off the boxes and circles, and used a filled dot and a circled filled dot to indicate the two different types of external legs corresponding to the different flavour symmetries.

The example in figure 6 involves also a bifundamental representation (which is a complex representation for $N > 2$).

As for the SU(2) theories, the above examples are particular members of a larger class of theories, denoted $\mathcal{T}_{(n,g)}(A_{N-1})$ [2]. This class of theories arise from the six-dimensional A_{N-1} (2,0) theory [9] compactified on $C \times \mathbb{R}^4$ where C is a genus g Riemann surface with n punctures. The punctures are due to codimension 2 defects filling $\mathbb{R}^4$ and intersecting C at points. An important difference compared to the SU(2) case is that now there is more than one type of possible defects. It was argued in [2] that the different codimension 2 defects are classified by partitions of N (which can be represented graphically in terms of Young tableaux). The outcome is that one can associate a Young tableau to each puncture. In the above examples we only encountered two kinds of punctures.

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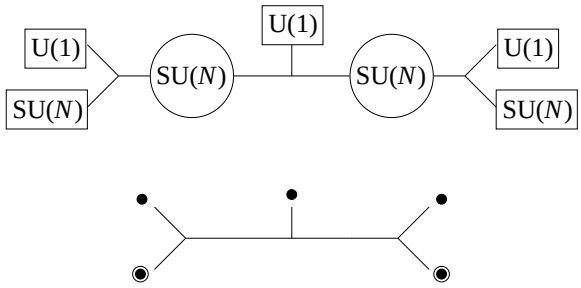

Figure 6. Another example of an SU(N) quiver.

As for the SU(2) quivers, the genus of the Riemann surface depends on the number of loops in the quiver diagram. The first quiver in figure 5 corresponds to a four-punctured sphere that has degenerated as in figure 4, where each of the two spheres has two punctures, one of each kind. Other distributions of the punctures lead to more exotic descriptions; the different possibilities are related by S-duality, the prototypical example being [26]. To construct more general quivers, the basic building block is the theory associated with three generic punctures on a sphere. See [2] for further details.

3.3 The relation between the two

In order to find a relation between the A_{N-1} Toda theories and the SU(N) quiver theories we need to find a rule that associates to a puncture described by a certain Young diagram a corresponding primary V_α in the Toda theory. In other words, we need a map from the Young tableau to some α. We propose that this map is essentially the same as the one used in [2] to associate a set of masses with a puncture. In particular this means (cf. section 4.2 in [2]) that a full puncture is associated with an unconstrained α whereas the puncture with U(1) flavour symmetry maps to an α of the form

$$\bullet \quad \mapsto \quad \varkappa\omega_1 \quad \text{or} \quad \varkappa\omega_{N-1}\,. \tag{3.21}$$

Here ω_1 (ω_{N-1}) is the highest weight of the fundamental (antifundamental) representation of SU(N). To see this we note that in the convention where the SU(N) root space is spanned by vectors whose components sum to zero, the weights of the fundamental representation of SU(N) can be chosen to be

$$\lambda_i = -u_i + \frac{1}{N}\sum_{j=1}^{N} u_j\,, \tag{3.22}$$

where u_i is the vector whose ith entry is 1 with all other components equal to 0 (note that $\sum_{i=1}^{N}\lambda_i = 0$). The highest weight is

$$\omega_1 = \lambda_1 = \frac{1}{N}(1{-}N, 1, \ldots, 1)\,, \tag{3.23}$$

and the highest weight of the antifundamental representation (with weights $\bar\lambda_i = -\lambda_{N+1-i}$) is

$$\omega_{N-1} = -\lambda_N = \frac{1}{N}(1, \ldots, 1, 1{-}N)\,. \tag{3.24}$$

Note that in this notation the simple roots are $e_i = u_i - u_{i+1}$ $(i = 1, \ldots, N-1)$ and the positive roots are $u_i - u_j$ $(i < j)$.

Even before doing any calculations we can perform some consistency checks of our proposal. It is not too difficult to see that the cases in which we have a perturbative description in terms of a conventional gauge theory quiver precisely correspond to the correlation functions for which the $\mathcal{W}$ algebra ambiguities in higher-point correlation functions are absent. This fact is quite encouraging and supports our proposal.

Additional evidence for the rule (3.21) can be obtained by considering the perturbative contributions. The one-loop matter contribution to the prepotential in the SU(N) theory with $2N$ (anti)fundamentals is proportional to

$$\sum_{\lambda} \sum_{\varrho \in \text{reps}} (\langle a, \lambda \rangle + \mu_\varrho)^2 \left[\log(\langle a, \lambda \rangle + \mu_\varrho) - \frac{3}{2} \right] . \tag{3.25}$$

Here the first sum runs over the weights λ_i of the representation ϱ and the second sum runs over the various representations ϱ of the gauge group that the matter fields transform in. In the SU(N) case the different ϱ comprise $2N$ (anti)fundamental representations. Decomposing the flavour symmetry as $\mathrm{SU}(N)^2\mathrm{U}(1)^2$ it is natural to take N masses associated with the matter fields in the fundamental representation of SU(N) (gauge) to transform in the fundamental of (flavour) SU(N) and similarly for the remaining N matter fields. We use the conventions

$$\mu_i = \frac{\kappa}{N} - \langle m, \lambda_i \rangle \,, \tag{3.26}$$

and

$$\bar{\mu}_i = -\frac{\bar{\kappa}}{N} + \langle \bar{m}, \lambda_i \rangle \,, \tag{3.27}$$

where λ_i are the weights (3.22). Note that $\sum_i \lambda_i = 0$, so $m/\bar{m}$ only contains $N-1$ independent parameters.

The above decompositions means that (3.25) can be written as:

$$\sum_{i,j=1}^{N} \left(\langle a, \lambda_i \rangle - \frac{\kappa}{N} + \langle m, \lambda_j \rangle \right)^2 \left[\log(\langle a, \lambda_i \rangle - \frac{\kappa}{N} + \langle m, \lambda_j \rangle) - \frac{3}{2} \right] \tag{3.28}$$

$$+ \sum_{i,j=1}^{N} \left(\langle a, \lambda_i \rangle - \frac{\bar{\kappa}}{N} + \langle \bar{m}, \lambda_j \rangle \right)^2 \left[\log(\langle a, \lambda_i \rangle - \frac{\bar{\kappa}}{N} + \langle \bar{m}, \lambda_j \rangle) - \frac{3}{2} \right] . \tag{3.29}$$

Note that since both the gauge and flavour groups involve the fundamental representation of SU(N) both sums run over the same set of weights. This means that the gauge parameters a and the mass parameters m, $\bar{m}$ are treated on the same footing. This is required for the identification with a CFT correlation function and provides evidence for the identification of the pieces involving κ and $\bar{\kappa}$ with the U(1) punctures.

Including also the ϵ dependence using the rules (A.6), the perturbative contribution to Z in the SU(N) theory with $2N$ fundamentals can be written (note that $\prod_{i<j}(a_i - a_j) = \prod_{e>0}\langle a, e\rangle$).

$$\frac{\prod_{e>0} \Upsilon(\langle a, e \rangle - \epsilon_1)\Upsilon(\langle a, e \rangle - \epsilon_2)}{\prod_{i,j} \Upsilon(\langle a, \lambda_i \rangle - \frac{\kappa}{N} + \langle m, \lambda_j \rangle)\Upsilon(\langle a, \lambda_i \rangle - \frac{\bar{\kappa}}{N} + \langle \bar{m}, \lambda_j \rangle - \epsilon)} \,. \tag{3.30}$$

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Comparing this to the contribution in the Toda theory four-point function:

$$\langle\alpha_1|V'_{\varkappa}|\alpha\rangle\langle\alpha|V'_{\bar{\varkappa}}|\alpha_4\rangle\,, \tag{3.31}$$

using (3.14) we find that if we redefine the primaries as

$$\mathcal{V}_\alpha = \frac{\left[\pi\mu\gamma(b^2)b^{2-2b^2}\right]^{\langle\alpha,\rho\rangle/b}}{\prod_{e>0}\Upsilon(\langle Q-\alpha,e\rangle)}V_\alpha\,, \qquad \mathcal{V}'_{\varkappa} = \frac{\left[\pi\mu\gamma(b^2)b^{2-2b^2}\right]^{\langle\varkappa,\rho\rangle/b}}{\Upsilon(\varkappa)}V'_{\varkappa}\,, \tag{3.32}$$

the two expressions agree up to the expected Vandermonde determinant $\prod_{i<j}(a_i-a_j)^2$, and a factor that only depends on b, provided the parameters on the two sides are identified as in (3.34) below. In (3.32) (and elsewhere) we are using a slight abuse of notation, using $\varkappa$ to denote both the vector (3.21) and the coefficient in front of ω_1/ω_{N-1} in that vector. It should be clear from the context which is meant. As in the Liouville case it seems possible that the difference between

$$\int\prod_i \mathrm{d}a_i \prod_{i<j}(a_i-a_j)^2\,|Z|^2 \tag{3.33}$$

and the four-point function can be removed by a minor redefinition of Z.

It is fairly straightforward to extend this discussion to any conventional quiver such as for instance the one in figure 6. Such linear quivers correspond to correlation functions of the type (3.20). Correlation functions on the torus involving a string of $V'_{\varkappa}$'s correspond to conventional necklace quivers and can therefore also be studied.

Next we would like to evaluate the instanton corrections on the gauge theory side and compare them to the chiral blocks in the Toda theory. In (A.3) the one-instanton correction in the SU(N) theory with $2N$ fundamentals is given and in (A.14) we have calculated the chiral block in the A_2 Toda theory. Using (3.11) and (3.12), it can be shown that (A.14) agrees with (A.3) provided we factor out a $(1-q)^{\kappa(\bar{\kappa}-\epsilon)/3}$ term from Z_{inst} and identify[8]

$$z=-q\,,\quad \alpha_1=m+Q\,,\quad \varkappa=\kappa+3Q\,,\quad \bar{\varkappa}=\bar{\kappa}+6Q\,,\quad \alpha_4=\bar{m}+Q\,,\quad \alpha=a+Q\,. \tag{3.34}$$

This represents a highly non-trivial test of the proposed relation.

We should stress that, if the conjectured relation is correct, the instanton counting gives closed expressions for any correlation function in the class (3.20) in the s channel. Such expressions would represent a new result. Note also that the intermediate states/descendants on the CFT side are labelled by sets of partitions and there may be a more direct correspondence with the partitions appearing in the instanton counting method. Finally, we mention that one can also discuss the Seiberg-Witten curve along the lines of [1] by replacing the stress tensor with a more general $\mathcal{W}$ algebra current.

[8]There appears to be more than one possible identification, which is probably due to the symmetry under the Weyl group. Also, there are some differences depending on which choice is made in (3.21) and which weights are used.

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4 Discussion and outlook

In this paper we argued that the connection proposed in [1] between the Liouville theory in two dimensions and $4d$ $\mathcal{N} = 2$ SU(2) quiver theories, extends to a connection between the $2d$ A_{N-1} Toda theories and the class of $4d$ $\mathcal{N} = 2$ SU(N) quiver theories studied in [2]. Although we have only performed selected tests of this idea, the agreement is nevertheless quite striking.

It is clearly important to perform further tests of the suggested relations. One possible approach is the following. It has been shown that when some of the α_i take certain special values, the four-point correlation functions (and also higher-point functions) in the Liouville [17, 27] and Toda theories [22, 23, 28] satisfy certain differential equations. These equations can be solved exactly in terms of special functions with explicit integral representations. On the gauge theory side it might be possible to sum up all instanton contributions and make contact with the results in [17, 22, 23, 27, 28]. However, perhaps the best approach is to show that the integral of the absolute square of the instanton partition function in the quiver gauge theory satisfies the same differential equation as the corresponding Toda theory correlation function. As the prepotential is known to satisfy Picard-Fuchs type equations such an approach does not seem unreasonable.

Let us also mention that there exists a conjectured relation in the Liouville theory between a certain four-point correlation function on the sphere and a one-point function on the torus (see (3.30) in [17]). It may be possible to check this proposal using the gauge theory approach.

It would also be nice to find more examples of relations between $2d$ CFTs and $4d$ quiver gauge theories. There is a Toda theory associated with any Lie algebra. Are these related to quiver gauge theories?

The most basic question is what the underlying reason for the Toda/quiver connection is. Hopefully it will be possible to use the quiver gauge theories to learn more about the Toda theories and vice versa.

Acknowledgments

The main part of this work was done while visiting Instituto Superior Técnico, Lisbon whose hospitality is appreciated. This work was supported by a grant from the Swedish Science Council.

A Nekrasov instanton counting and $\mathcal{W}$ algebra chiral blocks

In this appendix we collect some facts about instanton counting and $\mathcal{W}$ algebras.

A.1 Nekrasov Instanton counting

The instanton counting method was developed by Nekrasov [12] (further details can be found in [13, 29]; see also [30]). Below we focus on theories with a simple gauge group. More general quiver theories can also be treated in a similar manner. In particular, bifundamental matter has been treated in [1, 31].

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The instanton partition function can be written

$$Z_{\text{inst}} = \sum_k q^k Z_k \,, \tag{A.1}$$

where for SU(N)

$$Z_k = \frac{1}{k!} \int \prod_{i=1}^{k} \frac{\mathrm{d}\varphi_i}{2\pi i} \,, z_k(a, \varphi, \mu; \epsilon) \,. \tag{A.2}$$

Here z_k depends on the field content of the model. The integrals in (A.2) can in many cases be performed explicitly leading to closed expressions for the Z_k. We will not give the details here. Instead we only give one example: the one instanton contribution in the SU(N) theory with $2N$ fundamentals can be written

$$Z_1 = \frac{1}{\epsilon_1 \epsilon_2} \sum_{i=1}^{N} \frac{M(\hat{a}_i)}{\prod_{j \neq i} (\hat{a}_i - \hat{a}_j)(\hat{a}_i - \hat{a}_j + \epsilon)} \,, \tag{A.3}$$

where $\epsilon \equiv \epsilon_1 + \epsilon_2$, $\sum_{i=1}^{N} \hat{a}_i = 0$ and

$$M(x) = \prod_{i=1}^{N} (x - \mu_i)(x + \bar{\mu}_i - \epsilon) \,. \tag{A.4}$$

It is convenient to write $\hat{a} = \sum_{i=1}^{N-1} a_i \, e_i$ where e_i are the simple roots of the A_{N-1} Lie algebra. In the particular cases of SU(2) and SU(3), this translates into $\hat{a} = (a, -a)$ and $\hat{a} = (a_1, -a_1 + a_2, -a_2)$, respectively. In the case of SU(2) we also write for the masses:

$$\mu_1 = m_1 + \kappa_1 \,, \qquad \mu_2 = m_2 + \kappa_2 \,, \qquad \bar{\mu}_1 = m_1 - \kappa_1 \,, \qquad \bar{\mu}_2 = m_2 - \kappa_2 \,. \tag{A.5}$$

In addition to the instanton contribution there is also a perturbative (1-loop) piece, Z^{pert}, in the full partition function. Z^{pert} is a product of various factors. For SU(N) it is obtained from the building blocks (note that there is some freedom to redefine the μ_i and $\bar{\mu}_i$ by shifts):

$$\begin{aligned}
z^{1-\text{loop}}_{\text{gauge}}(\hat{a}) &= \prod_{i<j} \exp[-\gamma_{\epsilon_1,\epsilon_2}(\hat{a}_i - \hat{a}_j - \epsilon_1) - \gamma_{\epsilon_1,\epsilon_2}(\hat{a}_i - \hat{a}_j - \epsilon_2)] \,, \\
z^{1-\text{loop}}_{\text{fund}}(\hat{a}, \mu) &= \prod_{i} \exp[\gamma_{\epsilon_1,\epsilon_2}(\hat{a}_i - \mu)] \,, \\
z^{1-\text{loop}}_{\text{antifund}}(\hat{a}, \bar{\mu}) &= \prod_{i} \exp[\gamma_{\epsilon_1,\epsilon_2}(\hat{a}_i + \bar{\mu} - \epsilon)] \,, \\
z^{1-\text{loop}}_{\text{adjoint}}(\hat{a}, \mu) &= \prod_{i<j} \exp[\gamma_{\epsilon_1,\epsilon_2}(\hat{a}_i - \hat{a}_j - \mu - \epsilon)] \,.
\end{aligned} \tag{A.6}$$

Here the function $\gamma_{\epsilon_1,\epsilon_2}(x)$ (not to be confused with $\gamma(x)$ defined in (2.11)) is related to $\Gamma_2(x|\epsilon_1, \epsilon_2)(x)$ defined in (2.8), (2.9) as

$$\gamma_{\epsilon_1,\epsilon_2}(x) = \log \Gamma_2(x|\epsilon_1, \epsilon_2) \,. \tag{A.7}$$

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In the case when the gauge group is Sp($2N$) the Z_k's are determined by

$$Z_k = \frac{(-1)^k}{2^n n!} \int \prod_{i=1}^{n} \frac{\mathrm{d}\varphi_i}{2\pi i}\, z_n(a,\varphi,m;\epsilon)\,, \tag{A.8}$$

where $n = \lfloor \frac{k-1}{2} \rfloor$ and $\lfloor\cdot\rfloor$ denotes the integer part. We have[9]

$$z_k^{\mathrm{fund}}(a,\varphi,m;\epsilon) = (m-\epsilon/2)^{k-2n} \prod_{i=1}^{n} ((m-(\epsilon/2))^2 - \varphi_i^2)\,, \tag{A.9}$$

and

$$\begin{aligned} z_k^{\mathrm{gauge}}(a,\varphi;\epsilon) \;=\; & \frac{\epsilon^n}{\epsilon_1^n \epsilon_2^n} \left[\frac{1}{2\epsilon_1\epsilon_2 \prod_{l=1}^{N} ((\epsilon/2)^2 - a_l^2)} \prod_{i=1}^{n} \frac{\varphi_i^2(\varphi_i^2-\epsilon^2)}{(\varphi_i^2-\epsilon_1^2)(\varphi_i^2-\epsilon_2^2)} \right]^{k-2n} \\ & \times \prod_{i=1}^{n} \frac{1}{P(\varphi_i-\epsilon/2)P(\varphi_i+\epsilon/2)(4\varphi_i^2-\epsilon_1^2)(4\varphi_i^2-\epsilon_2^2)} \\ & \times \prod_{i<j} \frac{(\varphi_i-\varphi_j)^2((\varphi_i-\varphi_j)^2-\epsilon^2)(\varphi_i+\varphi_j)^2((\varphi_i+\varphi_j)^2-\epsilon^2)}{((\varphi_i-\varphi_j)^2-\epsilon_1^2)((\varphi_i-\varphi_j)^2-\epsilon_2^2)((\varphi_i+\varphi_j)^2-\epsilon_1^2)((\varphi_i+\varphi_j)^2-\epsilon_2^2)}\,. \end{aligned} \tag{A.10}$$

In the Sp($2N$) case there is no restriction on the a_i and the index takes the values $i = 1,\ldots,N$. It is not known how to write closed expressions for the above integrals for arbitrary instanton numbers (see [15] for a discussion), but at low orders in the instanton expansion the integrals can be explicitly performed.

In the Sp(2) theory the leading terms are (using a convenient redefinition of the masses)

$$Z_{\mathrm{inst}} = 1 - q\frac{(m_1^2-\kappa_1^2)(m_2^2-\kappa_2^2))}{(4a^2-\epsilon^2)} + \mathcal{O}(q^2) \tag{A.11}$$

Since SU(2) can also be viewed as Sp(2) this expression is an alternative to the one obtained viewing SU(2) as a subgroup of U(2).

A.2 $\mathcal{W}$ algebra chiral blocks

Here we illustrate how one calculates chiral blocks in a $\mathcal{W}$ algebra, using the $\mathcal{W}_3$ algebra as an example. At level 1 a convenient basis of descendants is

$$|\psi_1\rangle = L_{-1}|\alpha\rangle\,, \qquad |\psi_2\rangle = \left(W_{-1} - \frac{3w_\alpha}{2\Delta_\alpha} L_{-1}\right)|\alpha\rangle\,. \tag{A.12}$$

The 2×2 Gram matrix $\langle\psi_i|\psi_j\rangle$ then becomes

$$\begin{pmatrix} 2\Delta_\alpha & 0 \\ 0 & \Delta_\alpha\left[\frac{32}{22+5c}(\Delta_\alpha+\frac{1}{5})-\frac{1}{5}\right] - \frac{9}{2}\frac{w_\alpha^2}{\Delta_\alpha} \end{pmatrix}, \tag{A.13}$$

[9] These expressions differ slightly from the ones in [15] where the case $\epsilon = 0$ was the main focus. The prescription used here arises from the method in [16]

and the chiral block of the four-point function in the s-channel becomes

$$\begin{aligned}
&\frac{(\Delta_{\alpha_2}+\Delta_\alpha-\Delta_{\alpha_1})(\Delta_{\alpha_3}+\Delta_\alpha-\Delta_{\alpha_4})}{2\Delta_\alpha} \\
&+\left(\frac{w_{\alpha_3}}{2}-\frac{w_\alpha}{2}-w_{\alpha_4}+\frac{3}{2}\frac{\Delta_\alpha}{\Delta_{\alpha_3}}w_{\alpha_3}-\frac{3}{2}\frac{\Delta_{\alpha_4}}{\Delta_{\alpha_3}}w_{\alpha_3}-\frac{3}{2}\frac{\Delta_{\alpha_3}}{\Delta_\alpha}w_\alpha+\frac{3}{2}\frac{\Delta_{\alpha_4}}{\Delta_\alpha}w_\alpha\right) \\
&\times\left(\frac{w_{\alpha_2}}{2}-\frac{w_\alpha}{2}-w_{\alpha_1}+\frac{3}{2}\frac{\Delta_\alpha}{\Delta_{\alpha_2}}w_{\alpha_2}-\frac{3}{2}\frac{\Delta_{\alpha_1}}{\Delta_{\alpha_2}}w_{\alpha_2}-\frac{3}{2}\frac{\Delta_{\alpha_2}}{\Delta_\alpha}w_\alpha+\frac{3}{2}\frac{\Delta_{\alpha_1}}{\Delta_\alpha}w_\alpha\right) \\
&\times\left(\Delta_\alpha\left[\frac{32}{22+5c}\left(\Delta_\alpha+\frac{1}{5}\right)-\frac{1}{5}\right]-\frac{9}{2}\frac{w_\alpha^2}{\Delta_\alpha}\right)^{-1}. \qquad \text{(A.14)}
\end{aligned}$$

Note that the first term is simply the standard Virasoro chiral block at this level. At higher levels the expressions quickly become very involved.

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